Quantum Phase Transitions

Mapping d-space dimensions Quantum system to d+1-space dimensions Classical system.

The mapping we are referring to is nothing but the standard Feynman path integral representation of the propagator $\langle x | e^{-iHt} | x' \rangle$ where we will replace it by β ('imaginary time') and express the partition f^n $Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int_x \langle x | e^{-\beta H} | x \rangle$. of the quantum problem as the partition f^n of a classical problem.

As a refresher, consider single-particle QM $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ so that the propagator $\langle x | e^{-iHt} | x' \rangle$ equals

$$= \int_{x_1 \ldots x_N} \langle x | e^{-iH dt} | x_1 \rangle \langle x_1 | e^{-iH dt} | x_2 \rangle \langle x_2 | \ldots \langle x_N | e^{-iH dt} | x' \rangle \qquad (N dt = t)$$

$$= \int_{x_1 \ldots x_N} \langle x | e^{-i \frac{\hat{p}^2}{2m} dt} e^{-i V(\hat{x}) dt} | x_1 \rangle \ldots$$

$$= \int_{x_1 \ldots x_N} e^{im \left(\frac{x - x_1}{dt}\right)^2 dt} e^{-i V(x_1) dt}$$
$$e^{im \left(\frac{x_1 - x_2}{dt}\right)^2 dt} e^{-i V(x_2) dt}$$
$$\cdots$$
$$e^{im \left(\frac{x_N - x'}{dt}\right)^2 dt} e^{-i V(x') dt}$$

$$= \int_{\substack{x(t=0)=x \\ x(t)=x'}} Dx(t)\, e^{i\int_0^t dt'\left[\left(\frac{dx}{dt'}\right)^2 - V(x(t'))\right]}$$

By almost identical derivation, the partition f^n of the same quantum Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ is

$$Z = \mathrm{Tr}\, e^{-\beta H}$$

$$= \int_{x(0)=x(\beta)} D\,x(\tau)\, e^{-\int_0^\beta d\tau\left[\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau))\right]}$$

$$(it \to \tau)$$

Thus, the partition f^n of this zero-dimensional QM problem maps to the partition f^n of a one-dimensional classical problem in system size $\beta\ (= \frac{1}{T})$ with periodic boundary condition.

More generally, consider a quantum Hamiltonian of variables $\hat{p}(\vec{r})$, $\hat{\varphi}(\vec{r})$ which live in a d-dim space ($\vec{r} = r_1, \ldots, r_d$) with commutation relations $[\hat{\varphi}(\vec{r}), \hat{p}(\vec{r}')] = i\,\delta_{rr'}$.

Now the Hilbert space is a direct product of local Hilbert space corresponding to states acted upon by $\hat{p}(r), \hat{\varphi}(r)$ at $\vec{r}$: $\mathcal{H} = \otimes\, \mathcal{H}(r)$.

Let's assume that the allowed coordinates form a d-dim cubic lattice.

Let's consider the following Hamiltonian:

$$\hat{H} = \sum_{\hat{r}} \left[\frac{\hat{p}^2(r)}{2m} + V(\{\varphi(r), \nabla\varphi(r)\}) \right]$$

where $V(\{\hat{\varphi}(r), \nabla\hat{\varphi}(r)\})$ is some arbitrary term that depends on $\hat{\varphi}$ and its spatial derivatives.

Following the same strategy as above, the partition function of this system becomes,

$$Z = \int_{\substack{\varphi(r,0) \\ = \varphi(r,\beta)}} \mathcal{D}\varphi(r,\tau)\, e^{-\int_0^\beta d\tau\, d^dx \left[\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + V(\varphi, \nabla\varphi) \right]}$$

For example, if $V(\varphi, \nabla\varphi) = (\nabla\varphi)^2 + t\varphi^2 + u\varphi^4$,

then $Z = \int \mathcal{D}\varphi\, e^{-S}$ where S is now the action for the Ising model in $d+1$ dimensions, where the imaginary time direction is of size β.

As $\beta \to \infty$, one obtains a truly $d+1$-dimensional system. The critical point in this system is tuned via quantum fluctuations which get mapped to thermal fluctuations in one higher dimension via the above mapping.

An example: Transverse field Ising model in $d = 1$.

The Hamiltonian is given by

$$\hat{H} = \underbrace{-J}_{\text{Ising Coupling}}\sum_i \sigma_i^z \sigma_{i+1}^z \underbrace{-h}_{\text{Transverse field}} \sum_i \sigma_i^x$$

When $h = 0$, this is identical to 1d classical Ising model, which, as you recall, can be solved using Transfer matrix method. Also, it's crucial to note that the above model is very different from the 'longitudinal field Ising model',

$$\hat{H} = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^z$$

which <u>doesn't</u> have the $\sigma_i^z \to -\sigma_i^z$ symmetry

The Ising symmetry of the transverse field Ising model is a consequence of

$$U^\dagger H U = H$$

where $U = \prod_i \sigma_i^x$

Note that $U^\dagger U = 1$ and $U^2 = 1$

Also $U^{+} \sigma_i^z U = -\sigma_i^z$ and therefore if we add a longitudinal field $\sum_i \sigma_i^z$ to H, it will <u>lose</u> the Ising symmetry.

Since symmetries can be spontaneously broken, we next ask: does the ground state(s) of H have the Ising symmetry or not?

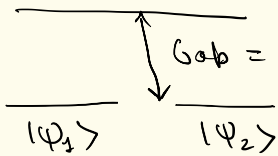
To address this question, we consider the two limits: $J \gg h$ and $J \ll h$.

① $J \gg h$

This is the familiar limit because in this limit the ground state(s) essentially behave like the classical 1d Ising model at $T=0$ ($h=0$). At $h=0$, there are two ground states:

$$|\psi_1\rangle = \prod_i |\uparrow\rangle_i \qquad |\psi_2\rangle = \prod_i |\downarrow\rangle_i$$

Therefore at $h=0$, the spectrum looks like:



The two ground states $|\varphi_1\rangle$, $|\varphi_2\rangle$ break the Ising symmetry spontaneously because unlike the Hamiltonian H, $|\varphi_1\rangle, |\varphi_2\rangle$ are not unchanged by the action of unitary $U = \prod_i \sigma_i^x$.

Specifically,
$$U|\varphi_1\rangle = |\varphi_2\rangle$$
$$U|\varphi_2\rangle = |\varphi_1\rangle$$

i.e. the two ground states interchange under the action of H.

Therefore, the Ising symmetry is spontaneously broken at $h = 0$.

What happens when $h \neq 0$ but still $h \ll J$?

The two ground states will mix with each other at N'th order in perturbation theory in h because one has to flip all spins to go from $|\varphi_1\rangle$ to $|\varphi_2\rangle$. Therefore, the degenerate perturbation theory will now yield the following two eigenstates,

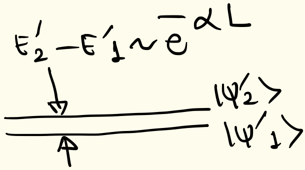
$$|\psi_1'\rangle = \frac{1}{\sqrt{2}}\left[|\psi_1\rangle + |\psi_2\rangle\right]$$

$$|\psi_2'\rangle = \frac{1}{\sqrt{2}}\left[|\psi_1\rangle - |\psi_2\rangle\right]$$

with the eigen-energy difference

$$E_2' - E_1' = \frac{h^L}{J^{L-1}} \propto h e^{-\alpha L}$$

where $\alpha \sim \log(J/h)$. Thus, the spectrum looks like



In the thermodynamic limit, $L \to \infty$, the two eigenstates are again degenerate.

But now one notices that <u>neither</u> $|\psi_1'\rangle$ and $|\psi_2'\rangle$ break the Ising symmetry spontaneously. Indeed, upto a phase, both $|\psi_1'\rangle$ and $|\psi_2'\rangle$ are invariant under the Ising symmetry

$$U|\psi_1'\rangle = |\psi_1'\rangle$$
$$U|\psi_2'\rangle = -|\psi_2'\rangle.$$

So does one conclude that the symmetry is not broken spontaneously when $h \neq 0$? No! This is a wrong argument!

Recall that to talk about spontaneous symmetry breaking, one has to include an infinitesimal symmetry breaking field $\Delta H = \epsilon \sum_i \sigma_i^z$ and consider the limit $L \to \infty$ first and then $\epsilon \to 0$.

When one introduces an infinitesimal longitudinal field $\epsilon \sum_i \sigma_i^z$, one obtains back the eigenstates $|\psi_1\rangle = \prod_i |\uparrow\rangle_i$ and $|\psi_2\rangle = \prod_i |\downarrow\rangle_i$, thus leading to spontaneous symmetry breaking. The transverse field $h \sum_i \sigma_i^x$ will modify $|\psi_1\rangle$ and $|\psi_2\rangle$ little bit (e.g. one of the ground states will look like

$$|\uparrow\uparrow\uparrow \ldots \uparrow\rangle + \frac{h}{J}\left[|\downarrow\uparrow\uparrow\uparrow \ldots \uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow \ldots \uparrow\rangle + \ldots\right]$$

) but

overall, the two low-lying states will be degenerate in the thermodynamic limit and will almost look like $\prod_i |\uparrow\rangle_i$ and $\prod_i |\downarrow\rangle_i$ in the presence of infinitesimal longitudinal field $\epsilon \sum_i \sigma_i^z$.

Therefore, the spontaneous symmetry breaking occurs in the limit $h \ll J$

Next, consider the limit $h \gg J$.

② $h \gg J$.

Let's first set $J = 0$. There is a unique ground state.

$$|\psi\rangle = \prod_i |\rightarrow\rangle_i \quad \text{where}$$

$|\rightarrow\rangle$ denotes a spin pointing along the $+x$ direction. This unique ground state is invariant under the unitary transformation $U = \prod_i \sigma_i^x$

$$\prod_i \sigma_i^x \prod_j |\rightarrow\rangle_j = \prod_j |\rightarrow\rangle_j$$

Therefore, symmetry is unbroken in the limit $h \gg J$.

Excitations around the two limits :

When $J \gg h$, the excitations are domain walls and cost energy $O(J)$.

$$|\uparrow\ \uparrow\ \uparrow \dots \uparrow \underbrace{\uparrow\ \ \downarrow}_{\text{domain wall}}\ \downarrow\ \downarrow\ \downarrow \dots\rangle$$

Domain wall is a 'topological defect', such topological defects are a hallmark of spontaneous symmetry breaking.

On the other hand, when $h \gg J$, the excitations are flipped spins from $+x$ direction to $-x$ direction and cost energy $O(h)$.

$$|\rightarrow\ \rightarrow\ \rightarrow\ \rightarrow\ \leftarrow\ \rightarrow\ \rightarrow\ \rightarrow\ \rightarrow\rangle$$

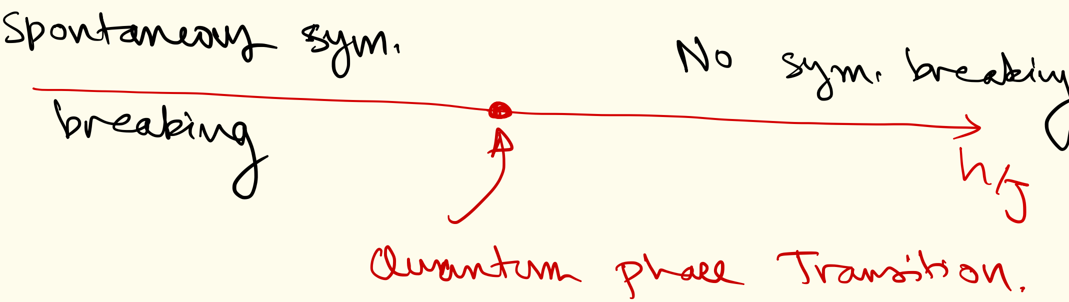
flipped spin.

This is <u>not</u> a topological defect.

Lets make a table that summarizes the differences between $h \gg J$ limit and $J \gg h$ limit.

$J \gg h$	$h \gg J$
1. Two degenerate ground states in the thermodynamic limit.	Unique ground state in the thermodynamic limit.
2. Ground states break the Ising symmetry spontaneously.	Ground state does <u>not</u> break the Ising sym. spontaneously.
3. Excitations are domain walls.	Excitations are individually flipped spins

Following our earlier discussion on phases and phase transitions, these two limits correspond to two different phases of matter and therefore must be separated by a phase transition where the ground state energy is a non-analytic function of the tuning parameter h/J



We next study this phase transition using <u>mean-field theory</u>.

Mean field Theory of Quantum Phase Transition in the 1d Transverse field Ising Model.

The actual Hamiltonian is :

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

Within the mean-field theory, we replace the interaction term (i.e. the J term) as.

$$\sigma_i^z \sigma_{i+1}^z \rightarrow \langle \sigma_i^z \rangle \sigma_{i+1}^z + \sigma_i^z \langle \sigma_{i+1}^z \rangle - \langle \sigma_i^z \rangle \langle \sigma_{i+1}^z \rangle$$

Further we assume a <u>homogeneous</u> solution $\langle \sigma_i^z \rangle = m \quad \forall i$.

The mean-field Hamiltonian becomes

$$H_{MF} = -2Jm\sum_i \sigma_i^z - h\sum_i \sigma_i^x + NJm^2$$

Were the coordination no. z (here z = 2), we would have

$$H_{MF} = -zJm\sum_i \sigma_i^z - h\sum_i \sigma_i^x + \frac{Nz}{2}Jm^2$$

The ground state energy is

$$\frac{E}{N} = -\sqrt{(zJm)^2+h^2} + \frac{Jzm^2}{2}$$

Doing a taylor expansion in m, one obtains,

$$\frac{E}{N} = -h \left[1 + \left(\frac{zJm}{h}\right)^2\right]^{1/2} + \frac{Jzm^2}{2}$$

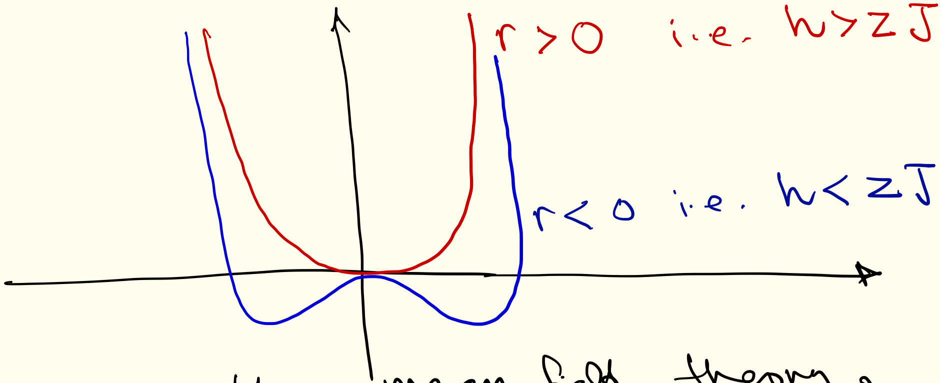
$$\simeq -h \left[1 + \frac{1}{2}\frac{(zJm)^2}{h^2} - \frac{1}{4}\left(\frac{zJm}{h}\right)^4 + \cdots\right] + \frac{Jzm^2}{2}$$

$$= r\,m^2 + u\,m^4 + \text{constant}$$

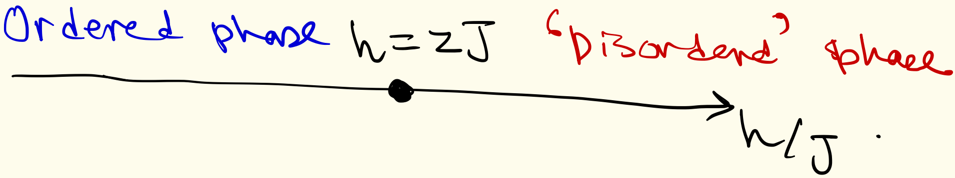
where $$r = \frac{Jz}{2} - \frac{(zJ)^2}{2h}$$

$$u = \frac{1}{4}\frac{(zJm)^4}{h^3}$$

This has exactly the same form as the Landau free energy for the Ising transition!



Thus, within the mean-field theory, the sym. is spontaneously broken when $h < zJ$.



An alternate method is to find m self consistently,

$$m = \langle \psi_0 | \sigma_i^z | \psi_0 \rangle$$ where

$|\psi_0\rangle$ is the mean-field ground state.

$$\langle \psi_0 | \sigma_i^z | \psi_0 \rangle = \frac{zJm}{\sqrt{(zJm)^2 + h^2}}$$

$$\Rightarrow \quad m = \frac{zJm}{\sqrt{(zJm)^2 + h^2}}$$

$$\Rightarrow (zJm)^2 + h^2 = (zJ)^2$$

$$\Rightarrow \quad m^2 = \frac{(zJ)^2 - h^2}{(zJ)^2}$$

$$\Rightarrow \quad m = \pm \sqrt{1 - \frac{h^2}{(zJ)^2}}$$

Thus, m vanishes above the critical ratio: $\left(\frac{h}{J}\right)_c = z$, as also seen above using the Landau theory.

Duality in 1+1-d Ising model.

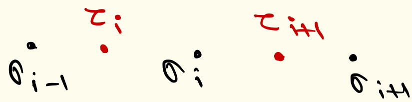
1+1-d transverse field Ising model admits a duality in the same spirit as the duality we earlier discussed for the O(2) model in 2d.

The Hamiltonian is

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

Consider the dual 'domain wall' variables, which live on the sites of the dual lattice (= links of the original lattice) defined as

$$\boxed{\begin{aligned} \sigma_i^z \sigma_{i+1}^z &= \tau_{i+1}^x \\ \sigma_i^x &= \tau_i^z \tau_{i+1}^z \end{aligned}}$$



The second relation may be inverted as

$$\tau_i^z = \prod_{j<i} \sigma_j^x$$

where the product $\prod_{j<i}$ is over a semi-infinite string.
Similarly, $\sigma_i^z = \prod_{j\leq i} \tau_j^x$. Therefore, the relations

are almost symmetrical. The 'physical meaning' of the dual variable τ_i^z is that it is a domain wall creation operator at site 'i' as obvious from its definition $\tau_i^z = \prod_{j<i} \sigma_j^x$.

Therefore, the dual variables are the 'disorder variables', very much like the vortices being the dual variables in the O(2) model.

Most importantly, in terms of the dual variables,

$$H = -J\sum_i \tau_i^x - h\sum_i \tau_i^z \tau_{i+1}^z$$

Thus, the role of J and h is flipped compared to the definition in terms of σ_i variables.

The SSB phase of σ variables corresponds to the paramagnet phase of the τ variables and vice-versa. The 'global information' such as the ground state degeneracy is lost under this duality, and is related to the fact that two configs. of σ: $\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow$ and $\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow$ correspond to the same configuration in terms of the domain wall variables τ.

Since $J \leftrightarrow h$ under duality, the critical point must satisfy $\frac{J}{h} = \frac{h}{J}$ i.e. $h = \pm J$.

Choosing $h = J$, at the critical point,

$$H = -J \left[\sum_i \sigma_i^z \sigma_{i+1}^z + \sum_i \sigma_i^x \right]$$

Since $\sigma_i^z = \prod_{j \leq i} \tau_j^x$, the duality has a very interesting consequence for the operator product expansion of 2d Ising model. Due to Ising symmetry, $\sigma^z \to -\sigma^z$, the OPE of the scaling variable φ ($\sim \sigma^z$) can't contain φ:

$$\varphi . \varphi \sim \mathbb{1} + \varphi^2$$

Since σ^z is related to τ^x via duality, and τ^x is the energy operator in the dual variables, the system has an emergent $\varphi^2 \to -\varphi^2$ symmetry. This implies that OPE of φ^2 with itself cannot contain φ^2, and only higher order terms such as φ^4 appear.

$$\varphi^2 . \varphi^2 \sim \mathbb{1} + \varphi^4.$$

Mapping of 1+1-d transverse field Ising model to 2+0 classical Ising model.

$$Z = \mathrm{Tr}\, e^{-\beta\left[-\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^x\right]}$$

$$= \mathrm{Tr}\, e^{N d\beta\left[\sum_i \sigma_i^z \sigma_{i+1}^z + h\sum_i \sigma_i^x\right]} \qquad N d\beta = \beta$$

$$\simeq \mathrm{Tr}\, \prod e^{d\beta \sum \sigma_i^z \sigma_{i+1}^z}\, e^{d\beta h \sum \sigma_i^x}$$

$$= \langle \sigma(\{i\}, \tau=0)|\, e^{d\beta \sum \sigma_i^z \sigma_{i+1}^z}\, e^{d\beta h \sum \sigma_i^x}\, |\sigma(\{i\}\tau=1)\rangle$$

$$\langle \sigma(\{i\}, \tau=1|\, e^{d\beta \sum \sigma_i^z \sigma_{i+1}^z}\, e^{d\beta h \sum \sigma_i^x}\, |\sigma\{i\}, \tau=2\rangle$$

Therefore, all we need is the matrix element

$$\langle \sigma |\, e^{d\beta\, h\, \sigma^x}\, |\sigma'\rangle.$$

Writing $\langle \sigma |\, e^{d\beta h \sigma^x}\, |\sigma'\rangle = A\, e^{B\sigma\sigma'}$

one finds $A = \frac{1}{2}\cosh(2 h d\beta)$

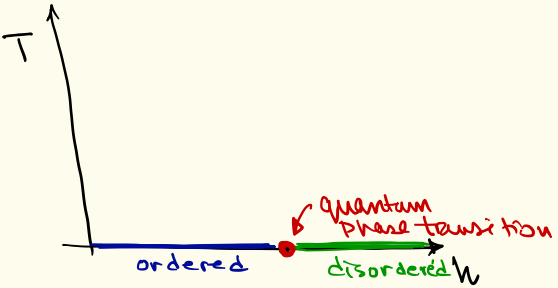
$$e^{-2B} = \tanh(h\, d\beta)$$

Thus,

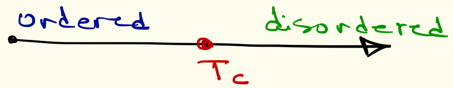
$$Z = e^{\sum_{x,\tau}\left[d\beta\; \sigma(x,\tau)\,\sigma(x+1,\tau) + B\,\sigma(x,\tau)\sigma(x,\tau+1)\right]}$$

which is the partition f^n of 2+0-d classical Ising model.

Relation between universal quantities of 1+1-d Ising and 2+0-d Ising.



1+1-d Ising Phase diagram



2+0-d Classical Ising

At $T=0$, the imaginary time correlations such as $\langle g.s.| \sigma^z(x,\tau)\, \sigma^z(0,0)\, |g.s.\rangle$ precisely correspond to the correlations $\langle \sigma(x,\tau)\, \sigma(0,0)\rangle$ in the classical Ising model. In the quantum model, the gap closes as $\Delta \sim (h-h_c)^{\nu}$ which precisely corresponds to the scaling of ξ^{-1} in the classical model (and of course in the quantum model as well).

At $T \neq 0$, the correspondence is more interesting: the quantum model maps to the classical model in a slab of size $L_x \times \beta$.

Lets consider the behavior of the correlation length in three regimes.

(i) $h_c - h \gg T$: In this regime, the system is at low-T on the ordered side. Using our intuition from the 1d <u>classical</u> Ising model, where we found the correlation length $\sim e^{\text{gap}/T}$, here again one expects $\xi \sim e^{\Delta/T}$ where $\Delta \sim h_c - h$.

(ii) $h - h_c \gg T$: In this regime, the system is at low-T on the disordered side. Since the correlation length is already finite even at $T=0$ in the disordered phase, one expects $\xi \sim \frac{1}{\Delta}$ where $\Delta \sim h - h_c$.

(iii) $T \gg |h - h_c|$: This is the regime where one may employ intuition from quantum-to-classical mapping. In the classical system, this corresponds to a system close to criticality where $L_y (= \beta) \ll \xi_\infty$. Therefore, the physical correlation length is limited by the finite size of the system and $\xi \sim L_y = \beta$.

Based on this discussion, it is convenient to draw the following phase diagram for the quantum system (see Sachdev's book for a comprehensive discussion)